Smooth Loops, Generalized Coherent States and Geometric Phases

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Abstract

A description of generalized coherent states and geometric phases in the light of the general theory of smooth loops is given.

Key words: Quasigroups, Coherent States, Geometric Phases

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1 INTRODUCTION

Berry (1984) showed that a quantum system whose Hamiltonian depends on some parameters λ^a and which slowly evolves in time in such a way that during the evolution the state of the system traces out a closed curve C in the space of these parameters, the wave function can get an additional geometrical phase $\gamma(C)$. This geometric phase depends on motion of the system in the space of parameters and independent of the dynamical evolution.

Later Aharonov and Anandan (1987) generalized Berry's result to any cyclic evolution of the quantum system by giving up the assumption of adiabaticity. Samuel and Bhandari (1988) introduced the geometric phase for

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an arbitrary case, the evolution of the quantum system need be neither unitary nor cyclic. These results have a simple geometric interpretation, the evolution of the geometric phase is determined by the natural connection in a fiber bundle over a space of rays (Simon, 1983; Aharonov and Anandan, 1987; Samuel and Bhandari, 1988).

Some years ago Giavarini and Onofri (1988) have described Berry's phase using generalized coherent states. Generalized coherent states are parametrized by points of homogeneous space where the group acts. These appear naturally in physical systems having dynamical symmetries, for instance, in certain nonstationary systems such as a quantum oscillator in a variable external field, or spin in a varying magnetic field (Perelomov, 1972, 1986).

In this paper we present an analysis of generalized coherent states and geometric from the theory of smooth loops (Sabinin, 1988, 1991, 1995) point of view.

2 SMOOTH LOOPS AND GEOMETRIC PHASES

Let G be an arbitrary Lie group and $T(g)$ its unitary irreducible representation acting in Gilbert space $\mathcal{H}$. Consider a fixed vector $|\psi_0\rangle \in \mathcal{H}$ and set of vectors (states) $\{|\psi_g\rangle\}$, where $|\psi_g\rangle = T(g)|\psi_0\rangle$.

Definition. (Perelomov, 1986) A system of states $\{|\psi_g\rangle : \ |\psi_g\rangle = T(g)|\psi_0\rangle\}$, where $g \in G$ and $T(g)$ is a representation of the group G, acting in the Hilbert space $\mathcal{H}$ ($|\psi_0\rangle$ is a fixed vector in this space), is called generalized coherent-state system $\{T, |\psi_0\rangle\}$.

Suppose $H \subset G$ is an isotropy subgroup for the vector $|\psi_0\rangle$, that as $T(h)|\psi_0\rangle = e^{\mathrm{i}\alpha(h)}|\psi_0\rangle, \quad \forall h \in H$. It shows that the coherent state $|\psi_g\rangle$ is determined by a point $x = x(g) = g \cdot H$ in the left coset space G/H. Choosing a representative element $g(x)$ in any equivalence class $x \in X = G/H$ one gets a set of generalized coherent states $\{|\psi_g\rangle : \ |\psi_g\rangle = e^{\mathrm{i}\alpha}|\psi_{g(x)}\rangle\}$. From the mathematical point of view we are now considering a certain left homogeneous space G/H uniquely determining the given coherent-state system. Actually $|\psi_g\rangle$ depends not on $g \in G$ but on the left coset $gH \in G/H$. Choosing one single element from any coset, one obtains in such a way a cross section $Q \subset H$, $Q \cap H = \{\text{single element}\}$, $Q \cap H = \{1_G\}$, $Q \cdot H = G$. Such a cross

section is called a *transversal* (*quasireductant*) of a homogeneous space G/H (see (Baer, 1940, 1942; Sabinin 1972)). Because of one-to-one correspondence between Q and G/H it gives us a parametrization of G/H by points of Q. We shall use this approach further on.

Any transversal (quasireductant) Q can be equipped in canonical way with the structure of a left loop $\langle Q,\ \star,\ \varepsilon\rangle$ (which means a set Q with the binary multiplication $\star$, right neutral element $\varepsilon,\ x\star\varepsilon = x$ and unique solvability of $a\star z = b,\ z = a\backslash b$). The construction is as follows: $\forall q_1, q_2 \in Q : q_1 \star q_2 = \pi_Q(q_1 \cdot q_2),\ \varepsilon = 1_G$, where $q_1 \cdot q_2$ means a product in the group G, and π_Q the projection from G onto Q along left cosets, $\{\pi_Q(q_1 \cdot q_2) = Q \cap [(q_1 \cdot q_2) \cdot H]\}$ (see (Sabinin, 1972) for details).

Remark. This construction frequently gives a two-sided loop $\langle Q,\ \star,\ \varepsilon\rangle$, which means that $a\star x = b,\ y\star c = d$ are uniquely solvable $(x = a\backslash b, y = d/c)$ and $\varepsilon \star x = x \star \varepsilon = x$.

Generally speaking the choice of transversal Q is not unique, although it is known how the structures of loops related to different reductants are connected (Sabinin, 1972). For a Lie group G with a nondegenerate Killing metric on Lie subgroup H a transversal (quasireductant) Q can be constructed in such a way that $h \cdot Q \cdot h^{-1} \subset Q$ $(\forall h \in H)$. Such Q is called a *reductant* (Sabinin,1972). The standard and unique way to construct a reductant is the following: taking $\mathfrak{g}$ and $\mathfrak{h}$ being Lie algebras for G and H, respectively, one can introduce a subspace $\mathfrak{m} = \{\zeta \in \mathfrak{g},\ \langle\zeta, \mathfrak{h}\rangle = 0\}$ (here $\langle\zeta,\eta\rangle$ means Killing's inner product on $\mathfrak{g}$). Since $\langle\zeta,\eta\rangle$ is nondegenerate on $\mathfrak{h}$ we get $\mathfrak{m} \cap \mathfrak{h} = \{0\}$, and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (direct sum). Taking $Q = \{q = \exp\xi, \xi \in \mathfrak{m}\}$, we get a smooth reductant, at least locally.

Thus for any $g \in G$ we have the unique decomposition $g = \pi_Q g \cdot \pi_H g = q\cdot h$, where $q = \pi_Q g \in Q,\ h = \pi_H g \in H$. Consequently, for the representation T of G we have

$$\begin{aligned} T(g) &= T(\pi_Q g \cdot \pi_H g) = T(\pi_Q g) \circ T(\pi_H g) \\ &= T(q) \circ T(h) \equiv D(q) \circ T(h), \quad (q \in Q,\ h \in H). \end{aligned} \tag{1}$$

(We have used $D(q)$ instead of $T(q)$ in order to emphasize that $D(q)$ is considered only for $q \in Q$.) Calculating, further, $T(q_1 \cdot q_2);\ q_1, q_2 \in G$, we find

$$D(q_1) \circ D(q_2) \equiv T(q_1) \circ T(q_2) = T(q_1 \cdot q_2)$$

$$= T(\pi_Q(q_1 \cdot q_2) \cdot \pi_H(q_1 \cdot q_2)) = T((q_1 \star q_2) \cdot \pi_H(q_1 \cdot q_2))$$
$$= D(q_1 \star q_2) \circ T(\pi_H(q_1 \cdot q_2)).$$

According to (Sabinin, 1972), $\pi_H(q_1 \cdot q_2)$ is uniquely determined by associator $l(q_1, q_2) = L^{-1}_{q_1 \star q_2} \circ L_{q_1} \circ L_{q_2}$, where $L_a b \stackrel{def}{=} a \star b$. Thus $T(\pi_H(q_1 \cdot q_2)) = \Lambda(q_1, q_2)$ can be regarded as an associator of our representation. As a result, we get

$$D(q_1 \star q_2) = D(q_1) \circ D(q_2) \circ \Lambda^{-1}(q_1, q_2), \ (q_1, q_2 \in Q). \tag{2}$$

Note that $D(\xi_1) \divideontimes D(\xi_2) = D(\xi_1 \star \xi_2)$ where $\divideontimes$ denotes the non-associative multiplication in the representation.

Let $|\psi_g\rangle = D(g)|\psi_0\rangle$ be an invariant state with respect to adjoint transformation: $|\psi_g\rangle = Ad_g D(q)|\psi_g\rangle$, where $Ad_g D(q) = D(g) \circ D(q) \circ D^{-1}(g)$, $g, q \in Q$. Using equation (2), we get

$$|\psi_g\rangle = D(g \star q) \circ \Lambda(g, q)|\psi_0\rangle. \tag{3}$$

For the infinitesimal transformations $g + dg = R_{\delta q} g$, where $R_{\delta q} g = g \star \delta q$ is the right action, we find (let $f(p)$ be a smooth mapping: $\mathfrak{M} \xrightarrow{f} \mathfrak{M}'$. We use f_* for the tangent mapping: $T_p(\mathfrak{M}) \xrightarrow{f_*} T_{f(p)}(\mathfrak{M})$)

$$d|\psi_g\rangle = dD(g)|\psi_0\rangle = D(g)(\Lambda(g, 0))_*(L_g^{-1})_* dg|\psi_0\rangle. \tag{4}$$

Let $g = g(t)$ be a curve in the space Q. Then equation (4) yields

$$(d/dt)|\psi_g\rangle = -D(g)(\Lambda(g, 0))_* D^{-1}(g)|\psi_g\rangle(L_g^{-1})_* dg/dt, \tag{5}$$

which is the differential equation for invariant state $|\psi_g\rangle$. Multiplying by $\langle\psi_g|$ we obtain

$$\langle\psi_g|(d/dt)|\psi_g\rangle = -\langle\psi_0|(\Lambda(g, 0)_*|\psi_0\rangle(L_g^{-1})_* dg/dt. \tag{6}$$

One can write (5) as

$$(d/dt)|\psi_g\rangle - \mathrm{i}A(t)|\psi_g\rangle = 0, \tag{7}$$

where $A = \mathrm{i}D(g)(\Lambda(g, 0))_* D^{-1}(g)(L_g^{-1})_* dg/dt$ is introduced. If one makes a gauge transformation

$$|\psi'_g\rangle = e^{\mathrm{i}\alpha(t)}|\psi_g\rangle,$$

then the A field transforms as

$$A' = A + d\alpha/dt,$$

i.e. as proper gauge potential. So equation (5) gives a definition of a parallel transport in the Hilbert space expressed in terms of the invariant states with respect to adjoint action of the loop Q.

Suppose that the normalized state $|\psi(t)\rangle = e^{\mathrm{i}\varphi(t)}|\psi_{g(t)}\rangle$ evolves according to the Schrödinger equation $\mathrm{i}(d/dt)|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$; hence $d\varphi/dt = -\langle\psi_g|\hat{H}|\psi_g\rangle + \mathrm{i}\langle\psi_g|(d/dt)|\psi_g\rangle$. For cyclic evolution of a quantum system, $|\psi_{g(\tau)}\rangle = |\psi_{g(0)}\rangle$, $g(\tau) = g(0)$, the total phase being $\varphi = \gamma - \delta$, where

$$\delta = -\int_0^\tau \langle\psi_g|\hat{H}|\psi_g\rangle dt$$

is the dynamical phase and

$$\gamma = \mathrm{i}\oint_C \langle\psi_g|d|\psi_g\rangle$$

is the Aharonov–Anandan (AA) geometric phase (Aharonov and Anandan, 1987). Using equation(6), we get

$$\gamma = -\mathrm{i}\oint_C \langle\psi_0|(\Lambda(g,0))_*|\psi_0\rangle(L_g^{-1})_* dg. \tag{8}$$

Hence γ defined by equation (8), is independent of φ and uniquely determined by the associator and the curve $C \in Q$. Thus we come to conclusion that the A-A geometric phase is originated by nonassociativity.

3 Examples

Now we consider two applications of the above theory to the generalized coherent states for the groups $SU(1,1)$ and $SU(2)$.

1. Group $SU(1,1)$. The group $SU(1,1)$ can be consider as a group of transformations of the complex plane $\mathbb{C}$. The action of this group is intransitive and the complex plane $\mathbb{C}$ is divided into three orbits: $\mathbb{C}_+ = \{z : |z| < 1\}$, $\mathbb{C}_- = \{z : |z| > 1\}$, $\mathbb{C}_0 = \{z : |z| = 1\}$. The Lie algebra corresponding to the group $SU(1,1)$ is spanned by generators $K_0, K_\pm$ with the commutation

relations: $[K_0, K_\pm] = K_\pm,\ [K_-, K_+] = 2K_0$. Let us now restrict ourselves to consideration of $\mathbb{C}_+$. The set of complex numbers $\xi, \eta \in \mathbb{C}_+$ with the operation $\star$ forms a two-parametric loop $QH(2)$ (Nesterov and Stepanenko, 1986; Nesterov, 1989, 1990)

$$L_\zeta\eta \equiv \zeta \star \eta = \frac{\zeta + \eta}{1 + \overline{\zeta}\eta}, \tag{9}$$

here $\overline{\zeta}$ is the complex conjugate number $(\zeta = x + \mathrm{i}y, \quad \overline{\zeta} = x - \mathrm{i}y)$. The associator $l(\zeta, \eta) = L^{-1}_{\zeta\star\eta} \circ L_\zeta \circ L_\eta$ on $QH(2)$ is determined by

$$l(\zeta, \eta) = \frac{1 + \zeta\overline{\eta}}{1 + \eta\overline{\zeta}} \tag{10}$$

and can be written also as $l(\zeta, \eta) = \exp(\mathrm{i}\alpha), \quad \alpha = 2\arg(1 + \zeta\overline{\eta})$. This loop is isomorphic to the geodesic loop of a two-dimensional Lobachevski space realized as the upper part of two-sheeted unit hyperboloid H^2 (Sabinin, 1991, 1995). The isomorphism is established by exponential mapping $\zeta = e^{\mathrm{i}\varphi}\tanh\frac{\tau}{2}$ where (τ, φ) are inner coordinates on H^2.

The group $SU(1,1)$ is non-compact and all its unitary irreducible representations are infinite-dimensional. We shall consider only a discrete representation, which is determined by a single number $k = 1, 3/2, 2, 5/2, \ldots$ and $K_0|k, \mu|\rangle = \mu|k, \mu|\rangle,\ \mu = k + m$, where m is an integer $(m \geq 0)$. The operators $D(\xi)$, determined as

$$D(\xi) = \exp(\xi K_+ - \overline{\xi}K_-), \quad \xi = -\frac{\tau}{2}e^{-\mathrm{i}\varphi}, \tag{11}$$

form a nonassociative representation of the loop $QH(2)$ with the multiplication law (see (2))

$$D(\xi_1) \ast D(\xi_2) = D(\xi_1) \circ D(\xi_2) \circ \exp(-\mathrm{i}\alpha K_0) \tag{12}$$

where $\alpha = -\mathrm{i}\ln(l(\zeta_1, \zeta_2))$ and $\zeta_1 = e^{\mathrm{i}\varphi_1}\tanh|\xi_1|,\ \zeta_2 = e^{\mathrm{i}\varphi_2}\tanh|\xi_2|$. The canonical set $\{|\zeta\rangle\}$ of coherent states, corresponding to the choice $|\psi_0\rangle = |k, k\rangle$, is (Perelomov, 1986)

$$|\zeta\rangle = (1 - |\zeta|^2)^k \exp(\zeta K_+)|\psi_0\rangle, \tag{13}$$

where $\zeta = e^{\mathrm{i}\varphi}\tanh\frac{\tau}{2}$. The infinitesimal operators in this representation are

$$\langle\zeta|K_0|\zeta\rangle = k\frac{1 + |\zeta|^2}{1 - |\zeta|^2}, \quad \langle\zeta|K_+|\zeta\rangle = 2k\frac{\overline{\zeta}}{1 - |\zeta|^2}.$$

Now let us compute the geometric phase γ using (8) and (12). We find

$$\gamma(C) = \langle\psi_0|K_0|\psi\rangle \oint_C \delta\alpha \tag{14}$$

where $\delta\alpha = \mathrm{i}\ln((1+\overline{\zeta}\delta\zeta)/(1+\zeta\delta\overline{\zeta})) = \mathrm{i}(\overline{\zeta}\delta\zeta - \zeta\delta\overline{\zeta})$. Applying (9), we obtain $\delta\zeta = d\zeta/(1-|\zeta|^2)$. This yields

$$\gamma(C) = \langle\psi_0|K_0|\psi_0\rangle \oint_C \delta\alpha = \mathrm{i}k \oint_C \frac{\overline{\zeta}d\zeta - \zeta d\overline{\zeta}}{1-|\zeta|^2} = -kA, \tag{15}$$

where A is the area of the hyperboloid's surface corresponding to the region bounded by the closed path $C \in \mathbb{C}_+$, with taking into account the equation $K_0|\psi_0\rangle = k|\psi_0\rangle$.

On the basis of equation (13) one can calculate γ directly:

$$\frac{d\gamma(C)}{dt} = \mathrm{i}\langle\zeta|(d/dt)|\zeta\rangle = \mathrm{i}k\frac{\overline{\zeta}d\zeta/dt - \zeta d\overline{\zeta}/dt}{1-|\zeta|^2} = -2k\frac{d\varphi}{dt}\sinh^2\frac{\tau}{2} \tag{16}$$

and the total phase is the same as in (15),

$$\gamma(C) = \mathrm{i}\oint_C \langle\zeta|d\zeta\rangle = -2k\oint_C \sinh^2\frac{\tau}{2}\, d\varphi = -kA. \tag{17}$$

The unit hyperboloid H^2 can be considered as as the phase manifold of the quantum parametrically excited oscillator (Perelomov, 1986). Hence (15) determines the AA geometric phase for cyclic evolution of this oscillator.

2. Group $SU(2)$. The consideration is similar to that used for $SU(1,1)$. The essential difference between the group $SU(2)$ and $SU(1,1)$ is that the first is compact and simply connected, while the second is neither. The Lie algebra of $SU(2)$ is spanned by generators $J_0, J_\pm$ with the standard commutation relations: $[J_0, J_\pm] = J_\pm$, $[J_-, J_+] = -2J_0$. Any unitary irreducible representation of the group $SU(2)$ is determined by nonnegative integer or half-integer j. In the space of representation $\mathcal{H}^j$ we shall use the canonical basis $|j,\mu\rangle$ $-j \le \mu \ge j$ of eigenvectors of the operator $J_0:$ $J_0|j,\mu\rangle = \mu|j,\mu\rangle$. The generalized coherent states correspond to points of the two-demensional sphere S^2 and the set of operators $\{D(\xi)\}$ is given by (Perelomov, 1986)

$$D(\xi) = \exp(\xi J_+ - \overline{\xi}J_-), \quad \xi = -\frac{\theta}{2}e^{-\mathrm{i}\beta} \tag{18}$$

where $\beta = \pi - \varphi$ and θ, φ are the usual spherical coordinates. This corresponds to the stereographic projection onto the complex plane $\mathbb{C}$ from the north pole of the sphere. Applying the operator $D(\xi)$ in its normal form,

$$D(\xi) = \exp(\zeta J_+) \exp(\eta J_0) \exp(-\overline{\zeta} J_-), \tag{19}$$

where $\zeta = e^{\mathrm{i}\varphi} \tan\frac{\theta}{2}, \quad \eta = \ln(1 + |\zeta|^2)$, to the state vector $|\psi_0\rangle = |j, -j\rangle$, one gets the set of coherent states (Perelomov, 1986)

$$\begin{gathered} |\zeta\rangle = (1 + |\zeta|^2)^{-j} \exp(\zeta J_+)|j, -j\rangle, \\ \langle\zeta|J_0|\zeta\rangle = -j\frac{1 - |\zeta|^2}{1 + |\zeta|^2}, \quad \langle\zeta|J_+|\zeta\rangle = 2j\frac{\overline{\zeta}}{1 + |\zeta|^2}. \end{gathered} \tag{20}$$

The sphere S^2 admits a natural quasigroup structure, namely, S^2 is a local two-parametric loop $QS(2)$ (Nesterov and Stepanenko, 1986; Nesterov, 1989, 1990)

$$L_\zeta \eta \equiv \zeta \star \eta = \frac{\zeta + \eta}{1 - \overline{\zeta}\eta} \tag{21}$$

where $\zeta, \eta \in \mathbb{C}$ and the isomorphism between points of the sphere and the complex plane $\mathbb{C}$ is established by the stereographic projection from the north pole of the sphere: $\zeta = e^{\mathrm{i}\varphi} \tan\frac{\theta}{2}$. The associator is determined by

$$l(\zeta, \eta) = \frac{1 - \zeta\overline{\eta}}{1 - \eta\overline{\zeta}} \tag{22}$$

and can be written also as $l(\zeta, \eta) = \exp(\mathrm{i}\alpha), \quad \alpha = 2\arg(1 - \zeta\overline{\eta})$. The operators $D(\xi)$ form a nonassociative representation of $QH(2)$:

$$D(\xi_1) \circledast D(\xi_2) = D(\xi_1) \circ D(\xi_2) \circ \exp(-\mathrm{i}\alpha K_0) \tag{23}$$

where $\alpha = -\mathrm{i}\ln(l(\zeta_1, \zeta_2))$ and $\zeta_1 = e^{\mathrm{i}\varphi_1} \tan|\xi_1|, \ \zeta_2 = e^{\mathrm{i}\varphi_2} \tan|\xi_2|$.

Let us compute γ using (8). Applying (23), we find

$$\gamma(C) = \langle\psi_0|J_0|\psi_0\rangle \oint_C \delta\alpha \tag{24}$$

where $\delta\alpha = \mathrm{i}\ln((1 - \overline{\zeta}\delta\zeta)/(1 - \zeta\delta\overline{\zeta})) = \mathrm{i}(\overline{\zeta}\delta\zeta - \zeta\delta\overline{\zeta})$. From (21) we get $\delta\zeta = d\zeta/(1 + |\zeta|^2)$. Now taking into account $J_0|\zeta\rangle = -j|\zeta\rangle$, we obtain

$$\gamma(C) = \langle\psi_0|J_0|\psi_0\rangle \oint_C \delta\alpha = \mathrm{i}j \oint_C \frac{\overline{\zeta}d\zeta - \zeta d\overline{\zeta}}{1 + |\zeta|^2} = -j\Omega, \tag{25}$$

where Ω is the solid angle corresponding to the contour C. Using (20) and the definition

$$\gamma = \mathrm{i}\oint_C \langle\zeta|d|\zeta\rangle,$$

one can calculate γ directly. This yields the same result $\gamma = -j\Omega$.

Actually, the sphere S^2 can be considered as the phase manifold of the spin (Perelomov, 1986) and, consequently, equation (25) gives the AA geometric phase for the cyclic evolution of the spin. For instance, the spin precession in a variable magnetic field $\mathbf{H}(t)$ is described by the Hamiltonian $\hat{H} = -\kappa\mathbf{H}(t)\mathbf{J}(t)$ and the geometric phase $\gamma = -j\Omega$ (Berry, 1984).

4 CONCLUDING REMARKS

The above discussion shows that the generalized coherent states actually are detemined by points of the corresponding smooth loop Q and AA geometric phase is originated by nonassociativity of the operation (multiplication) in Q. Our approach can be applied also to evolution of a quantum system, that neither unitary nor cyclic. In this general case when the state vector does not return to the initial ray, the method of comparing the states is given by Pancharatnam (1975). Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be any two states which are not mutually orthogonal. The Pancharatnam phase β is defined as

$$e^{\mathrm{i}\beta} = \frac{\langle\psi_1|\psi_2\rangle}{\|\langle\psi_1|\psi_2\rangle\|}. \tag{26}$$

Now define

$$|\psi(\tau)\rangle = e^{\mathrm{i}\gamma(\tau)}|\psi(0)\rangle = D(g(\tau))|\psi_0\rangle$$

such that $\gamma(1) = \beta,\ |\psi(0)\rangle = |\psi_1\rangle,\ |\psi(1)\rangle = |\psi_2\rangle$. Then β is given by a line integral

$$\begin{aligned} \beta = -\mathrm{i}\int\langle\psi|d|\psi\rangle &= -\mathrm{i}\int_{g_0}^{g_1}\langle\psi_0|(\Lambda(g,0))_*|\psi_0\rangle(L_g^{-1})_*dg \\ &= -\mathrm{i}\int_{g_0}^{g_1}\langle\psi_1|D(g_1)(\Lambda(g,0))_*D^{-1}(g_1)|\psi_1\rangle(L_g^{-1})_*dg \end{aligned} \tag{27}$$

where we set $g_1 = g(1),\ g_2 = g(2)$.

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References

Aharonov Y. and Anandan J. (1987). *Physical Review Letters*, **58**, 1593.

Baer R. (1940). *Transactions of the American Mathematical Society*, **46**, 110.

Baer R. (1941). *Transactions of the American Mathematical Society*, **47**, 435.

Berry M. V. (1984). *Proceedings of the Royal Society A*, **392**, 45.

Giavarini G. and Onofri E. (1988). Generalized Coherent States and Berry's Phase, UTF Preprint.

Nesterov A. I. (1989).*Methods of Nonassociative Algebra in Physics,* Dr. Sci Theses, Institute of Physics, Estonian Academy of Science, Tartu.

Nesterov A. I. (1990). In *Quasigroups and Nonassociative Algebras in Physics*, Transactions of the Institute of Physics of the Estonian Academy of Science, **66**, 107.

Nesterov A. I. and Stepanenko V. A. (1986). On Methods of Nonassociative Algebra in Geometry and Physics, L. V. Kirensky Institute of Physics, Preprint 400F.

Pancharatnam S. (1975). *Proceedings of the Indian Academy of Sciience A*, **44**, 247.

Perelomov A. M. (1972). *Communications in Mathematical Physics*, **26**, 22.

Perelomov A. M. (1986). *Generalized Coherent States and Their Applications*, Springer-Verlag, Berlin Heidelberg, New York.

Sabinin L. V (1970). *Soviet Mathematical Doklady*, **13**, 970.

Sabinin L. V. (1972). *Mathematical Notes*,**12**, 799.

Sabinin L. V. (1988). In *Proceedings of Seminar on Vector and Tensor Analysis*, **23**, 133, Moscow Univ, Moscow.

Sabinin L. V. (1991). *Analytic Quasigroups and Geometry*, Friendship of Nations University, Moscow.

Sabinin L. V. (1995). *Russian Mathematical Survey*, **49**, 172.

Samuel J. and Bhandari R. (1988). *Physical Review Letters*, **60**, 2339.

Simon B. (1983). *Physical Review Letters*, **51**, 2167.